Simple linear regression - extended derivation of OLS solution

Fraida Fund

Set up

We assume a linear model

$$\hat{y}_i = w_0 + w_1 x_i$$

Given the (convex) loss function

$$L(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} [y_i - (w_0 + w_1 x_i)]^2$$

to find the minimum, we take the partial derivative with respect to each parameter, and set it equal to zero:

$$\frac{\partial L}{\partial w_0} = 0, \frac{\partial L}{\partial w_1} = 0$$

Solution for intercept w_0

First, let's solve for the intercept w_0. Using the chain rule, power rule:

$$\frac{\partial L}{\partial w_0} = \frac{1}{n} \sum_{i=1}^{n} (2)[y_i - (w_0 + w_1 x_i)](-1) = -\frac{2}{n} \sum_{i=1}^{n} [y_i - (w_0 + w_1 x_i)]$$

(We can then drop the constant factor when we set this expression equal to 0.)

Then, setting $\frac{\partial L}{\partial w_0} = 0$ is equivalent to setting the sum of residuals to zero:

$$\sum_{i=1}^{n} e_i = 0$$

(where e_i is the residual term for sample i).

Solution for slope w_1

Next, we work on the slope:

$$\frac{\partial L}{\partial w_1} = \frac{1}{n}\sum_{i=1}^{n} 2[y_i - (w_0 + w_1 x_i)](-x_i)$$

$$\implies -\frac{2}{n}\sum_{i=1}^{n} x_i[y_i - (w_0 + w_1 x_i)] = 0$$

Again, we can drop the constant factor. Then, this is equivalent to:

$$\sum_{i=1}^{n} x_i e_i = 0$$

(where e_i is the residual term for sample i).

Solving two equations for two unknowns

From setting the $\frac{\partial L}{\partial w_0} = 0$ and $\frac{\partial L}{\partial w_1} = 0$ we end up with two equations involving the residuals:

$$\sum_{i=1}^{n} e_i = 0, \sum_{i=1}^{n} x_i e_i = 0$$

where

$$e_i = y_i - (w_0 + w_1 x_i)$$

We can expand $\sum_{i=1}^{n} e_i = 0$ into

$$\sum_{i=1}^{n} y_i = n w_0 + \sum_{i=1}^{n} x_i w_1$$

then divide by n, and we find the intercept

$$w_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - w_1 \frac{1}{n}\sum_{i=1}^{n} x_i$$

i.e.

$$w_0^* = \bar{y} - w_1 \bar{x}$$

where $\bar{x}, \bar{y}$ are the means of x, y.

To solve for w_1, expand $\sum_{i=1}^n x_i e_i = 0$ into

$$\sum_{i=1}^n x_i y_i = \sum_{i=1}^n x_i w_0 + \sum_{i=1}^n x_i^2 w_1$$

and multiply by n.

$$n \sum_{i=1}^n x_i y_i = n \sum_{i=1}^n x_i w_0 + n \sum_{i=1}^n x_i^2 w_1$$

Also, multiply the “expanded” version of $\sum_{i=1}^n e_i = 0$,

$$\sum_{i=1}^n y_i = n w_0 + \sum_{i=1}^n x_i w_1$$

by $\sum x_i$, to get

$$\sum_{i=1}^n x_i \sum_{i=1}^n y_i = n \sum_{i=1}^n x_i w_0 + (\sum_{i=1}^n x_i)^2 w_1$$

Now, we can subtract to get

$$n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i = n \sum_{i=1}^n x_i^2 w_1 - (\sum_{i=1}^n x_i)^2 w_1$$

$$= w_1 \left(n \sum_{i=1}^n x_i^2 - (\sum_{i=1}^n x_i)^2 \right)$$

and solve for w_1^*:

$$w_1^* = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$